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Periodic and almost periodic solutions of conservation laws: global existence and decay

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Abstract. In this paper we survey recent results on the decay of periodic and almost periodic solutions of conservation laws. We also recall some recent results on the global existence of periodic solutions of conservation laws systems which lie in BV_{loc} and are constructed through Glimm scheme. The latter motivates a discussion on a possible strategy for solving the open problem of the global existence of periodic solutions of the Euler equations for nonisentropic gas dynamics. We base our decay analysis on a general result about space-time functions which are almost periodic in the space variable, established here for the first time. This result is an abstract version of Theorem 2.1 in [31], which in turn is an extention of the combined result given by Theorems 3.1-3.2 in [9].

Keywords: conservation laws, parabolic equations, almost periodic functions.

Mathematical subject classification: Primary 35L65; Secondary 35B35, 35B40.

1 Introduction

In this paper we survey recent results on the decay of periodic and almost periodic solutions of conservation laws. We also recall some recent results on the global existence of periodic solutions of conservation laws systems which lie in BV_{loc} and are constructed through Glimm scheme. The latter motivates a discussion on a possible strategy for solving the open problem of the global existence of periodic solutions of the Euler equations for nonisentropic gas dynamics. We base our decay analysis on a general result about space-time functions which are almost periodic in the space variable, established here for the first time. This result is an abstract version of Theorem 2.1 in [31], which in turn is an extention of the combined result given by Theorems 3.1-3.2 in [9].

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The study of the asymptotic behavior of the solutions of nonlinear conservation laws goes back to the pioneering paper of E. Hopf [40] on the Burgers equation, which started the modern analytical theory of conservation laws and may be seen as its second major landmark after the foundational 1860 paper of Riemann [52]. In the referred paper Hopf introduces the vanishing viscosity method which means to add an artificial viscosity to the original equation, solve the approximating equation, and then send the viscosity coefficient to zero. By means of a tricky tranformation of the dependent variables, the now called Hopf-Cole transfomation, which transforms the viscous Burgers equation into the heat equation, he was able to obtain an explicit formula for the solutions. This was then used to prove the convergence of the vanishing viscosity solutions and also provided an explicit formula for the solution of the inviscid equation. The work of Hopf was followed by a series of papers of Oleinik surveyed in [51] establishing existence and uniqueness of solutions of scalar conservation laws in one space variable with strictly convex flux function, which satisfy an admissible (entropy) condition on the points of discontinuity introduced by her. Oleinik's entropy condition was not only crucial for the uniqueness of the solutions but alone can explain the asymptotic behavior of such solutions in two important representative cases: periodic and compact supported initial data (see [56]). However, the problem of the asymptotic behavior of the entropy solutions of scalar conservation laws with strictly convex flux function was first solved and to a large extent by Lax, in his well known paper [44]. Therein, Lax considered the general class of initial data $u_0 \in L^\infty(\mathbb{R})$ satisfying the condition that the limit

$$M(u_0) = \lim_{L\to\infty} \frac{1}{L} \int_a^{a+L} u_0(x)\,dx$$

exists uniformly in $a \in \mathbb{R}$. This includes the two special cases mentioned above. For this general class of initial data, Lax proves the decay of the solution in the L^∞ norm to $M(u_0)$ as $t \to \infty$. His analysis is heavily based in an explicit formula for the solution found by him, motivated by Hopf's formula. The decay property for such general class of initial data is still unknown for flux functions which are not strictly convex. Also, as far as the author knows, the same general result is not known for the corresponding viscous equation!

Concerning periodic initial data, an important progress was achieved by Glimm and Lax in their influencial paper [38]. Therein, they prove the global existence of entropy solution of the Cauchy problem for a general class of strictly hyperbolic genuinely nonlinear 2×2 systems of conservation laws, for L^∞ initial data of small oscillation. The solutions are constructed through Glimm scheme and the regularization property is also shown to be a consequence of stronger estimates for the interaction of waves holding for 2×2 systems, proved by Glimm in his celebrated paper [37]. For periodic initial data, they prove that the solution so obtained decays in the L^∞ norm at a rate $O(t^{-1})$. More recently, the study of the asymptotic structure of general periodic $BV_{\rm loc}$ entropy solutions of systems in the same class considered by Glimm-Lax, possessing the same decay property, was analized in detail by Dafermos [20], using his method of generalized characteristics. For scalar conservation laws in two space variables with $BV_{\rm loc}$ periodic initial data and a nonlinearity condition on the flux functions, the decay of the periodic entropy solutions in the $L^1_{\rm loc}$ norm was proved by Engquist and E [28].

In [9], Chen and Frid establish a connection between the decay of periodic entropy solutions, $u(x,t)$, in the $L^1_{\rm loc}(\mathbb{R}^d)$ metric as $t \to \infty$ and the pre-compactness in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ of the associated scaling sequence $u^T(x,t) = u(Tx, Tt)$, $T > 0$. Here and in what follows we denote the L^p spaces with no reference to the range $\mathbb{R}^n$. They show that the pre-compactness of u^T in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ implies the decay in $L^1_{\rm loc}(\mathbb{R}^d)$ of those solutions, as $t \to \infty$. With the help of compactness results, such as those based on the compensated compactness theory (*e.g.* [24], [25], [7], [46], [47], [41], [8], [14], [35], etc.) and the one based on the kinetic formulation for scalar conservation laws in several space variables in [45], it was possible to obtain the decay in $L^1_{\rm loc}$ as $t \to \infty$ of large L^∞ periodic entropy solutions of many among the most representative systems of the theory, including, in particular, the Euler equations for isentropic gas dynamics, nonlinear elasticity and scalar conservation laws in several space variables with flux functions satisfying a nonlinearity condition. The result was also applied to obtain the decay of periodic solutions of systems of conservation laws with relaxation, in connection with results in [16], [17] also based on the compensated compactness theory. On the other hand, for *viscous* systems of conservation laws which are endowed with a strictly convex entropy, the decay of periodic solutions is in general easier and may be obtained by usual energy estimates as, for instance, those obtained in [39].

In [32] a periodic version of the Glimm scheme is presented, applicable to special classes of 2×2 systems for which it is available a simplication first noticed by Nishida [49] and further extended by Bakhvalov [1] and DiPerna [26]. For these special classes of 2×2 systems of conservation laws the simplification of the Glimm scheme gives global existence of solutions of the Cauchy problem with large initial data in $L^\infty \cap BV_{loc}(\mathbb{R})$, for Bakhvalov's class, and in $L^\infty \cap BV(\mathbb{R})$, in the case of DiPerna's class. It may also happen that the system is in Bakhvalov's class only at a neighboorhood $\mathcal{V}$ of a constant state, as it was proved for the isentropic gas dynamics by DiPerna [27], in which case the initial data is taken in $L^\infty \cap BV(\mathbb{R})$ with TV $(U_0) <$ const., for some constant which is $O((\gamma-1)^{-1})$ for the isentropic gas dynamics systems, $\gamma > 1$. For periodic initial data, our periodic formulation establishes that the periodic solutions so constructed, $u(\cdot, t)$, are uniformly bounded in $L^\infty \cap BV([0, \ell])$, for all $t > 0$, where ℓ is the period. We then obtain the asymptotic decay of these solutions by applying the theorem of Chen and Frid [9] combined with a compactness theorem of DiPerna [24]. The question about the decay of Nishida's solution was proposed by Glimm-Lax [38] and remained open since then. The classes to which the methods of [32] apply include the p-systems with $p(v) = \gamma v^{-\gamma}$, $-1 < \gamma < +\infty$, $\gamma \neq 0$, which, for $\gamma \geq 1$, model isentropic gas dynamics in Lagrangian coordinates. The results in [32] motivate a discussion on a possible strategy for the solution of the longstanding problem of the global existence of periodic solutions of the 3×3 Euler equations in gas dynamics.

In [31] the main result in [9] is extended to allow the study of the decay of generalized almost periodic solutions of inviscid and viscous conservation laws. Therein, several applications are presented including inviscid and viscous scalar conservation laws in several space variables, some inviscid systems in chromatography and gas dynamics, as well as many viscous 2×2 systems such as those of nonlinear elasticity and Eulerian isentropic gas dynamics, with artificial viscosity, among others. In the case of inviscid scalar equations and chromatography systems, the class of initial data for which decay results are proved includes, in particular, the L^∞ generalized limit periodic functions. As remarked in [31], following a procedure similar to the one in [9], the discusion about existence and decay of almost periodic solutions can be transported to the relaxation approximations. This extension becomes specially easier for the semilinear or kinetic approximations. In connection with semilinear and kinetic approximations we mention the recent L^∞ stability (of constant states) and compactness results of D. Serre [54].

2 A general result for space-time functions almost periodic in the space variable

In this section we recall some basic facts about almost periodic functions and establish a general result on space-time functions which are almost periodic in the space variable at each time.

The theory of almost periodic functions was founded by H. Bohr [5], in the context of continuous functions, and further extended to the context of measurable L^p_{loc} functions by Stepanoff [58], Wiener [63], Weyl [62], Besicovitch [2] and Besicovitch-Bohr [3] (see also [30]). For a complete account of this theory we refer also to the books of Bohr [4], Besicovitch [2] and Favard [29]. Here we will use a generalized concept of almost periodic functions which was introduced independently by Wiener and Stepanoff in the just referred papers.

Almost periodic functions were introduced by H. Bohr [4] in the context of continuous functions defined in the real line. According to the original definition, a function $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{C}$) is called *almost periodic* if, given $\varepsilon > 0$, there exists a number $l_\varepsilon > 0$, called inclusion interval with respect to ε, such that for all $x_0 \in \mathbb{R}$ there exists a number τ, with $x_0 \le \tau \le x_0 + l_\varepsilon$, called an ε-almost period or translation number with respect to ε, such that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \le \varepsilon.$$

If $f, g : \mathbb{R} \to \mathbb{R}$ are almost periodic functions, then $|f|$, $f + g$, fg are almost periodic, and so is g^{-1} if $|g(x)| > \delta > 0$, for all $x \in \mathbb{R}$. Also, the limit (mean value of f)

$$M_x(f) = \lim_{L\to\infty} \frac{1}{L} \int_a^{a+L} f(x)\,dx,$$

exists uniformly with respect to $a \in \mathbb{R}$.

The first fundamental theorem of the theory developed by H. Bohr asserts that any almost periodic function admits a unique represention by means of a Fourier series. In case f is real valued (which we assume henceforth) this may be represented by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos \lambda_n x + b_n \sin \lambda_n x),$$

where $a_n, b_n \in \mathbb{R}$ and $\lambda_n \in \mathbb{R} - \{0\}$. The coefficients a_0, a_n, b_n, $n = 1, 2, \dots,$ are given by

$$a_0 = M_x(f), \quad a_n = M_x(f(x)\cos \lambda_n x), \quad b_n = M_x(f(x)\sin \lambda_n x).$$

Moreover, Parseval identity holds:

$$M_x(|f|^2) = |a_0|^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

The second fundamental theorem asserts that the almost periodic functions can be uniformly approximated by trigonometric polynomials, that is, finite linear combinations of functions of the form $\sin \lambda x$, $\cos \lambda x$, with $\lambda \in \mathbb{R}$.

The definition and all the properties of the almost periodic functions in $\mathbb{R}$ can be immediately extended to continuous functions of several variables, $f:\mathbb{R}^d\to\mathbb{R}$. So, denoting $I_x^K=[x_1,x_1+K]\times\ldots\times[x_d,x_d+K]$, $K>0$, a continuous function $f:\mathbb{R}^d\to\mathbb{R}$ is said to be almost periodic if, given $\varepsilon>0$, there exists a number $l_\varepsilon>0$, called inclusion interval, such that for all $x_0\in\mathbb{R}^n$ one can find a vector $\tau\in I_{x_0}^{l_\varepsilon}$, called an ε-almost period, such that

$$\sup_{x\in\mathbb{R}^d}|f(x+\tau)-f(x)|\le\varepsilon.$$

A particular subclass of the almost periodic functions is that of the *limit-periodic functions*, that is, those continuous functions which can be uniformly aproximated by continuous periodic functions.

The concept of almost periodic functions was generalized by Stepanoff [58], Wiener [63], Weyl [62] and Besicovitch [2]. According to a definition due to Stepanoff and Wiener, used in this paper, a function $f:\mathbb{R}^d\to\mathbb{R}$ (or $f:\mathbb{R}^d\to\mathbb{C}$) in $L^1_{\rm loc}(\mathbb{R}^d)$ is a *generalized almost periodic function*, or briefly S-$a.p.$ function, if, given $\varepsilon>0$, there exists a number $l_\varepsilon>0$, still called inclusion interval, such that for all $x_0\in\mathbb{R}^d$, there exists a vector $\tau\in I_{x_0}^{l_\varepsilon}$, called ε-almost period, such that

$$\sup_{x\in\mathbb{R}^d}\int_{I_x}|f(y+\tau)-f(y)|\,dy\le\varepsilon,$$

where $I_x=I_x^1$. The absolute values and sums of S-$a.p.$ functions are S-$a.p.$ functions. For S-$a.p.$ functions $f,g\in L^\infty(\mathbb{R}^d)$, it is again true that their product is a S-$a.p.$ function. Also, in the case of a single variable, the unique representation by means of a Fourier series holds for S-$a.p.$ functions in general, and Parseval identity is true for S-$a.p.$ functions belonging to $L^\infty(\mathbb{R})$. The fundamental property of the S-$a.p.$ functions is that they can be approximated by trigonometrical polynomials in the metric d_S given by

$$d_S(f,g)=\sup_{x\in\mathbb{R}^d}\int_{I_x}|f(y)-g(y)|\,dy.$$

In particular, the limit (mean value of f)

$$M_x(f)=\lim_{L\to\infty}\frac{1}{L^d}\int_{I_a^L}f(x)\,dx,$$

exists uniformly in $a\in\mathbb{R}^d$, and so we may also write

$$M_x(f)=\lim_{L\to\infty}\frac{1}{(2L)^d}\int_{|x|_\infty\le L}f(x)\,dx,$$

where $|x|_\infty = \max_{1\le j\le d}|x_j|$. The extention of all the above concepts to vector valued functions is trivial.

Now, we consider a space-time bounded measurable vector valued function $\psi : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^m$, which is S-$a.p.$ in the space variable $x \in \mathbb{R}^d$. We say that $\psi(x,t)$ is S-$a.p.$ in x locally uniformly with respect to the time variable $t \in [0,\infty)$, if whenever $\tau \in \mathbb{R}^d$ is an ε-almost period of $\psi(x,t)$, for certain $t > 0$, then τ is an ε-almost period of $\psi(x,s)$, for all $s \in [0,t]$. For $T > 0$ we use the notation

$$\psi^T(x,t) = \psi(Tx, Tt). \tag{2.1}$$

Theorem 2.1. *Let $\psi : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^n$ be a bounded measurable function which is S-$a.p.$ in the space variable $x \in \mathbb{R}^d$, locally uniformly in $t \in [0,\infty)$. Let $l_\varepsilon(t)$ denote the inclusion interval with respect to ε of $\psi(x,t)$. Suppose:*

(i) $l_\varepsilon(t)/t \to 0$ as $t \to \infty$;

(ii) For $t, L > 0$,

$$\left| \frac{1}{(2L)^d}\int_{|x|_\infty<L} \psi(x,t)\,dx - \frac{1}{(2L)^d}\int_{|x|_\infty<L} \psi(x,0)\,dx \right| \le \mathcal{O}\left(\frac{t}{L}\right), \tag{2.2}$$

where $\mathcal{O}(s)$ is a function such that $\mathcal{O}(s) \to 0$ as $s \to 0$;

(iii) $\psi^T(x,t)$ is sequentially pre-compact in $L^1_{loc}(\mathbb{R}^d \times [0,\infty))$ as $T \to \infty$.

Then,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T M_x(|\psi(x,t) - \bar{\psi}|)\,dt = 0, \tag{2.3}$$

where $\bar{\psi} = M_x(\psi(x,0))$. Moreover, if there exists a continuous function $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$, satisfying

$$c_1|u - v|^2 \le \alpha(u,v) \le c_2|u - v|^2, \tag{2.4}$$

for u, v in a bounded set $K \subseteq \mathbb{R}^n$, for certain positive constants c_1, c_2, depending only on K, such that, for any $v \in \mathbb{R}^m$ and $c > 0$,

$$\frac{d}{dt}\int_{|\xi|<c} \alpha(\psi(\xi t, t), v)\,d\xi \le \frac{C}{t}, \tag{2.5}$$

for some $C > 0$, in the sense of the distributions over $(0,\infty)$, we have

$$\lim_{t\to\infty} M_x(|\psi(x,t) - \bar{\psi}|) = 0. \tag{2.6}$$

Proof. Let T_k be a subsequence of T going to infinity such that $\psi^{T_k}(x,t)$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+)$ to a certain L^∞ function $\bar\psi(x,t)$. We will show that $\bar\psi(x,t)=\bar\psi$, a.e. in $\mathbb{R}^{d+1}_+$, where $\bar\psi$ is given in the statement of the theorem.

1. We first show that, for almost all $t>0$, $\bar\psi(x,t)$ is independent of x. For that, given $\varepsilon>0$ and $t_0>0$, we consider the set

$$Q_{\varepsilon,t_0}=\{\frac{\tau}{T_k}\ :\ \tau \text{ is an } \varepsilon\text{-almost period of } \psi(\cdot,T_kt_0)\}.$$

We notice that Q_{ε,t_0} is dense in $\mathbb{R}^d$. This is clear from the fact that, since $l_\varepsilon(T_kt_0)/T_k\to 0$ as $T_k\to\infty$, given any cube with edge of lenght $\delta>0$, if $l_\varepsilon(T_kt_0)/T_k<\delta$, we can find a vector $\tau/T_k\in Q_{\varepsilon,t_0}$ inside this cube. We will show that for any $y\in\mathbb{R}^d$ we have $\bar\psi(x+y,t)=\bar\psi(x,t)$, for a.e. $x\in\mathbb{R}^d$. Now, let $\phi(x,t)$ be any continuous function with compact support contained in $[-L_0,L_0]^d\times[0,t_0]$, and let $y\in\mathbb{R}^d$ be given. By passing to a subsequence if necessary, we can find $y_k\in Q_{\varepsilon,t_0}$ such that $y_k\to y$ and y_k is an ε-almost period of $\psi^{T_k}(x,t)$ for $0\le t\le t_0$. We then have,

$$\begin{aligned}\int_{\mathbb{R}^{d+1}_+}\bar\psi(x+y,t)\phi(x,t)\,dxdt &= \lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}\psi^{T_k}(x+y,t)\phi(x,t)\,dxdt\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}\psi^{T_k}(x,t)\phi(x-y,t)\,dxdt\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}\psi^{T_k}(x,t)\phi(x-y_k,t)\,dxdt\\ &= \lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}\psi^{T_k}(x+y_k,t)\phi(x,t)\,dxdt\\ &\le \lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}\psi^{T_k}(x,t)\phi(x,t)\,dxdt+\\ &\quad + C(L_0,t_0)\|\phi\|_\infty\varepsilon\\ &= \int_{\mathbb{R}^{d+1}_+}\bar\psi(x,t)\phi(x,t)\,dxdt+\\ &\quad + C(L_0,t_0)\|\phi\|_\infty\varepsilon,\end{aligned}$$

and similarly we get

$$\int_{\mathbb{R}^{d+1}_+}\psi(x+y,t)\phi(x,t)\,dxdt\ge\int_{\mathbb{R}^{d+1}_+}\bar\psi(x,t)\phi(x,t)\,dxdt-C(L_0,t_0)\|\phi\|_\infty\varepsilon,$$

where $C(L_0, t_0)$ is a positive constant depending only on L_0, t_0. Since $\varepsilon > 0$ is arbitrary, we get

$$\int_{\mathbb{R}^{d+1}_+} \bar{\psi}(x+y,t)\phi(x,t)\,dxdt = \int_{\mathbb{R}^{d+1}_+} \bar{\psi}(x,t)\phi(x,t)\,dxdt$$

The function ϕ being also arbitrary we finally get $\bar{\psi}(x+y,t) = \bar{\psi}(x,t)$ for any x such that (x,t) and $(x+y,t)$ are Lebesgue points of $\bar{\psi}$. In particular, since $y \in \mathbb{R}^d$ is arbitrary, we get $\bar{\psi}(x_1,t) = \bar{\psi}(x_2,t)$ whenever (x_1,t) and (x_2,t) are Lebesgue points of $\bar{\psi}(x,t)$ and so $\bar{\psi}(x,t) = \bar{\psi}(t)$, for a.e. $(x,t) \in \mathbb{R}^{d+1}_+$, for a certain bounded measurable function $\bar{\psi}(t)$ depending only on t.

2. Now, from (2.2) it follows that $\bar{u}(t) = \bar{u}$, for a.e. $t \geq 0$. Indeed, for a.e. $t \in (0,\infty)$,

$$\begin{aligned}
\bar{\psi}(t) &= \lim_{k\to\infty} \frac{1}{(2L)^d} \int_{|x|_\infty < L} \psi^{T_k}(x,t)\,dx \\
&= \lim_{k\to\infty} \frac{1}{(2LT^k)^d} \int_{|x|_\infty < LT^k} \psi(x, T_k t)\,dx \\
&\leq \lim_{k\to\infty} \frac{1}{(2LT^k)^d} \int_{|x|_\infty < LT^k} \psi(x,0)\,dx + \mathcal{O}(\frac{t}{L}) \\
&\leq \bar{\psi} + \mathcal{O}(\frac{t}{L}).
\end{aligned}$$

Similarly, we get

$$\bar{\psi}(t) \geq \bar{\psi} + \mathcal{O}(\frac{t}{L}).$$

Therefore, letting $L \to \infty$, we get the assertion. Hence, we arrived at $\psi^T \to \bar{\psi}$ in $L^1_{\text{loc}}(\mathbb{R}^{d+1}_+)$, as $T \to \infty$. The latter clearly implies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{|\xi|_\infty \leq c} |\psi(\xi t, t) - \bar{\psi}|\,d\xi dt = 0, \tag{2.7}$$

for any $c > 0$. So, (2.3) will follow from (2.13) below, letting $\varepsilon \to 0$.

3. Now, we turn to the last part of the statement. Let $\alpha(u,v)$ be as in the statement of the theorem, and, for any fixed $c > 0$, denote

$$Y(t) = \int_{|\xi|_\infty \leq c} \alpha(\psi(\xi t, t), \bar{u})\,d\xi.$$

By (2.5), $Y(t)$ is in $BV_{\text{loc}}(0,\infty)$ and satisfies

$$\frac{dY}{dt}(t) \le \frac{C}{t}, \tag{2.8}$$

as measures, for some $C > 0$. The above inequality, together with (2.7), which, from (2.4), clearly implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)\,dt = 0, \tag{2.9}$$

leads, as in the proof of Theorem 2.3 of [11], to the conclusion that

$$\operatorname{ess}\lim_{t\to\infty} \int_{|\xi|_\infty \le c} |\psi(\xi t, t) - \bar{\psi}|\,d\xi = 0, \tag{2.10}$$

for any $c > 0$. For the sake of completeness, we briefly outline the proof of the last assertion, *i.e.*, that (2.8), (2.9) imply (2.10). Applying, formally, the fundamental theorem of calculus to the function $(t - T/2)Y^2(t)$, between $T/2$ and T, we get

$$Y^2(T) = \frac{2}{T}\int_{T/2}^T Y^2(t)\,dt + \frac{4}{T}\int_{T/2}^T (t - T/2)\frac{dY(t)}{dt}Y(t)\,dt,$$

which, by (2.8), gives

$$Y^2(T) \le \frac{2}{T}\int_{T/2}^T Y^2(t)\,dt + \frac{4C}{T}\int_{T/2}^T Y(t)\,dt,$$

which, together with (2.9), since $Y(t)$ is uniformly bounded, gives $Y(t) \to 0$, which in turn implies (2.10). Since the last inequality can also be achieved for a mollification of $Y(t)$, the just given formal argument can easily become rigorous.

4. To prove the decay in terms of mean values, let us partition $\mathbb{R}^d$ in a net of d-dimensional cubes with edges of length $3l_\varepsilon(t)$ parallel to the axes. Denote by S_t the set of such cubes contained in $\{x \in \mathbb{R}^d : |x| \le ct\}$, for certain fixed $c > 0$. Clearly, for each $I \in S_t$ there is an ε-almost period τ_I such that $I - \tau_I \supset [0, 2l_\varepsilon(t)]^d$. Let $N(t)$ be the number of elements of S_t. Since, $l_\varepsilon(t)/t \to 0$ as $t \to \infty$, we have $N(t) > 0$ and $(2c)^d/2 \le 3^d N(t) l_\varepsilon(t)^d / t^d \le (2c)^d$, for t

sufficiently large. Hence, we have

$$
\begin{aligned}
\int_{|\xi|\le c} |\psi(\xi t, t) - \bar{\psi}|\, d\xi &= \frac{1}{t^d}\int_{|x|\le ct} |\psi(x,t)-\bar{\psi}|\, dx \\
&\ge \frac{1}{t^d}\sum_{I\in S_t}\int_I |\psi(x,t)-\bar{\psi}|\, dx \\
&\ge \frac{1}{t^d}\sum_{I\in S_t}\int_I |\psi(x-\tau_I,t)-\bar{\psi}|\, dx \\
&\quad - \frac{1}{t^d}\sum_{I\in S_t}\int_I |\psi(x,t)-\psi(x-\tau_I,t)|\, dx \qquad (2.11)\\
&\ge -\varepsilon\frac{[(3l_\varepsilon(t))^d]N(t)}{t^d} + \frac{(3l_\varepsilon(t))^d N(t)}{t^d}\frac{1}{(3l_\varepsilon(t))^d} \\
&\quad \int_{[0,2l_\varepsilon(t)]^d} |\psi(x,t)-\bar{\psi}|\, dx \\
&\ge -\varepsilon(2c)^d + \frac{(2c/3)^d}{2}\frac{1}{l_\varepsilon(t)^d}\int_{[0,2l_\varepsilon(t)]^d} |\psi(x,t)-\bar{\psi}|\, dx
\end{aligned}
$$

Now, we partition $\mathbb{R}^d$ in a net of cubes with edges of length $l_\varepsilon(t)$ parallel to the axes. For each such cube I' there exists an ε-almost period $\tau_{I'}$ such that $I' - \tau_{I'} \subseteq [0, 2l_\varepsilon]^d$. Hence, we get

$$
\begin{aligned}
M_x(|\psi(x,t)-\bar{\psi}|) &= \lim_{s\to\infty}\frac{1}{(2sl_\varepsilon(t))^d}\int_{[-sl_\varepsilon(t),sl_\varepsilon(t)]^d} |\psi(x,t)-\bar{\psi}|\, dx \\
&\le \lim_{s\to\infty}\frac{1}{(2sl_\varepsilon(t))^d}\sum_{I'\subseteq[-sl_\varepsilon(t),sl_\varepsilon(t)]^d}\int_{I'} |\psi(x-\tau_{I'},t)-\bar{\psi}|\, dx \\
&+ \lim_{s\to\infty}\frac{1}{(2sl_\varepsilon(t))^d}\sum_{I'\subseteq[-sl_\varepsilon(t),sl_\varepsilon(t)]^d}\int_{I'} |\psi(x,t)-\psi(x-\tau_{I'},t)|\, dx \\
&\le \frac{1}{l_\varepsilon(t)^d}\int_{[0,2l_\varepsilon(t)]^d} |\psi(x,t)-\bar{\psi}|\, dx + 2^d\varepsilon
\end{aligned} \qquad (2.12)
$$

So, from (2.11) and (2.12) we obtain

$$
\int_{|\xi|\le c} |\psi(\xi t,t)-\bar{\psi}|\, d\xi \ge \frac{(2c/3)^d}{2} M_x(|\psi(x,t)-\bar{\psi}|) - \frac{3(2c)^d}{2}\varepsilon, \qquad (2.13)
$$

which, together with (2.10), gives

$$
\operatorname{ess}\limsup_{t\to\infty} M_x(|\psi(x,t)-\bar{\psi}|) \le 3^{d+1}\varepsilon,
$$

and, since ε can be taken arbitrarily small, we conclude

$$\operatorname{ess}\lim_{t\to\infty} M_x(|\psi(x,t)-\bar{\psi}|)=0,$$

as desired. □

Remark 2.1. The assumption (i) in Theorem 2.1 in the applications is usually reduced to a restriction on the growth rate of $l_\varepsilon(0)$ as ε goes to 0. Therefore, it matters to know whether, given any apriori specified growth rate, it is always possible to find a non-periodic $S\text{-}a.p.$ function such that its inclusion intervals l_ε obey the prescribed growth rate as $\varepsilon\to 0$. We show that this is true by means of the following construction, which establishes that, given any decreasing sequence $\varepsilon_k\downarrow 0$, as $k\to\infty$, with

$$\sum_{j=k+1}^{\infty}\varepsilon_j\le\varepsilon_k,\qquad k=1,2,\dots,$$

it is possible to construct a classical (non-periodic) almost periodic function (actually, limit-periodic function) whose inclusion intervals satisfy $l_{\varepsilon_k}=2(3^k)$. This, in particular, shows that there exist (non-periodic) almost periodic functions whose inclusion intervals satisfy whatever growth rate as $\varepsilon\to 0$ one may wish to prescribe. The construction is trivial. One starts with an interval, for instance, $(-1,1)$, take a function ϕ_0 in $C_0(-1,1)$, and set $f=\phi_0$ in $(-1,1)$. Then, we take $\phi_{0-}\in C_0((-3,-1))$ and $\phi_{0+}\in C_0((1,3))$, such that $\|\phi_0(\cdot\pm 2)-\phi_{0\pm}\|_\infty<\varepsilon_1/2$, define $\phi_1=\phi_{0-}+\phi_0+\phi_{0+}$ and set $f=\phi_1$ in $(-3,3)$. Similarly, we take $\phi_{1-}\in C_0((-9,-3))$, $\phi_{1+}\in C_0((3,9))$, such that $\|\phi_1(\cdot\pm 6)-\phi_{1\pm}\|_\infty<\varepsilon_2/2$, define $\phi_2=\phi_{1-}+\phi_1+\phi_{1+}$ and set $f=\phi_2$ in $(-9,9)$. In this way we can define inductively f in the whole real line. Specifically, assuming that $f=\phi_k$ in $(-3^k,3^k)$ with $\phi_k\in C_0((-3^k,3^k))$, we take $\phi_{k-}\in C_0((-3^{k+1},-3^k))$ and $\phi_{k+}\in C_0((3^k,3^{k+1}))$, such that $\|\phi_k(\cdot\pm 2(3^k))-\phi_{k\pm}\|_\infty<\varepsilon_{k+1}/2$, define $\phi_{k+1}=\phi_{k-}+\phi_k+\phi_{k+}$ and set $f=\phi_{k+1}$ in $(-3^{k+1},3^{k+1})$. It is easy to see that the so constructed function is almost periodic (actually, limit periodic) and satisfies $l_{\varepsilon_k}=2(3^k)$. For instance, if one wants to have $(\log\varepsilon)^{-1}l_\varepsilon\to 0$ as $\varepsilon\to 0$, it suffices to choose, say, $\varepsilon_k=e^{-k(3^k)}$.

Remark 2.2. An important special case of Theorem 2.1 is the one in which $\psi(x,t)$ is periodic with period independent of t, that is, $\psi(x+p_i\mathbf{e}_i,t)=\psi(x,t)$, for certain positive numbers $p_1,\dots,p_n$, where $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is the canonical basis of $\mathbb{R}^n$. In this special case condition (i) is trivially satisfied.

3 Inviscid and viscous conservation laws

We consider a multidimensional viscous or inviscid system of conservation laws

$$\partial_t u + \sum_{k=1}^{d} \partial_{x_k} f^k(u) = \sum_{kl} \partial^2_{x_k x_l} a_{kl}(u), \qquad x \in \mathbb{R}^d, t > 0, \tag{3.1}$$

where $u(x,t) \in \mathcal{U} \subseteq \mathbb{R}^n$, for some open set $\mathcal{U}$, and $f^k, a_{kl} : \mathcal{U} \to \mathbb{R}^n$ are smooth functions, for which an initial condition has been prescribed

$$u(x,0) = u_0(x). \tag{3.2}$$

In the inviscid case, that is, when the viscosity coefficients $a'_{k,l}(u)$ are all identically null, equation (3.1) takes the form

$$\partial_t u + \sum_{k=1}^{d} \partial_{x_k} f^k(u) = 0, \tag{3.3}$$

A smooth function $\eta : \mathcal{U} \to \mathbb{R}$ is an entropy for (3.1) if there are smooth functions $q_k, b_{kl} : \mathcal{U} \to \mathbb{R}$, $k, l \in \{1, \dots, d\}$, called the associated entropy-fluxes and entropy-viscosities, respectively, such that

$$\nabla q_k = \nabla\eta \nabla f^k, \quad \nabla b_{kl}(u) = \nabla\eta\nabla a_{kl} \quad k, l \in \{1, \dots, d\}. \tag{3.4}$$

If η is strictly convex, (3.4) implies that the matrices ∇f^k are simultaneously symmetrizable by $\nabla^2\eta$ and, in particular, $\xi_1\nabla f^1 + \cdots + \xi_d \nabla f^d$ is diagonalizable, for any $(\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. The latter is the condition for the system (3.1) to be hyperbolic in the case where $a_{kl}(u) \equiv 0$, for all $k, l = 1, \dots, d$.

In this paper, we will only consider bounded measurable solutions, although the results hold also with slight adaptations in the more general case of L^p_{loc} solutions.

Definition 3.1. We say that $u \in L^\infty(\mathbb{R}^{d+1}_+)$ is an entropy solution (or simply a solution) of (3.1)-(3.2) if for any non-negative $\phi \in C_0^1(\mathbb{R}^{d+1})$ and for any convex entropy η, with associated entropy-fluxes and entropy-viscosities q^k, b_{kl}, $k, l = 1, \dots, d$, such that

$$\sum_{kl} v_k^\top \nabla^2\eta(u)\nabla a_{kl}(u) v_l \ge 0, \quad \text{for all } (v_1, \dots, v_d) \in (\mathbb{R}^n)^d, \tag{3.5}$$

one has

$$\iint_{\mathbb{R}^{d+1}_+} \{\eta(u)\phi_t + \sum q^k(u)\phi_{x_k} + \sum b_{kl}(u)\phi_{x_k x_l}\}\,dxdt + \int_{\mathbb{R}^d} \eta(u_0)\phi(x,0)\,dx \geq 0. \tag{3.6}$$

In the inviscid case, we have the usual entropy inequality

$$\iint_{\mathbb{R}^{d+1}_+} \{\eta(u)\phi_t + \sum q^k(u)\phi_{x_k}\}\,dxdt + \int_{\mathbb{R}^d} \eta(u_0)\phi(x,0)\,dx \geq 0. \tag{3.7}$$

As usual, since the coordinate functions and their opposites $\pi_{i\pm}(u) = \pm u_i$, $i = 1,\dots,n$, are obviously convex entropies with associated entropy-fluxes and entropy-viscosities $\pm f^k$, $\pm a_{kl}$, respectively, which trivially satisfy (3.5), the inequality (3.6) with $\eta(u) = \pi_{i\pm}(u)$, $i = 1,\dots,n$, implies that u is a weak solution of (3.1)-(3.2), i.e., the equation

$$\iint_{\mathbb{R}^{d+1}_+} \{u\phi_t + \sum f^k(u)\phi_{x_k} + \sum a_{kl}(u)\phi_{x_k x_l}\}\,dxdt + \int_{\mathbb{R}^d} u_0(x)\phi(x,0)\,dx = 0, \tag{3.8}$$

holds for any $\phi \in C_0^1(\mathbb{R}^{d+1})$. When $a_{kl}(u) \equiv 0$, $k,l = 1,\dots,d$, entropy solutions are in general non-smooth, which is a basic fact in the theory of conservation laws (see, *e.g.*, [21], [53], [56]).

4 Decay of periodic solutions of conservation laws

In this section we recall the main result of [9] on the decay of periodic solutions of hyperbolic conservation laws and mention some of its most important applications. We consider the Cauchy problem (3.3)-(3.2), for an inviscid multidimensional system of conservation laws. Given a solution $u(x,t)$ for this problem, the scaling sequence $u^T(x,t)$ is defined as in (2.1). So, let $u(x,t)$ be an entropy solution of (3.3)-(3.2), periodic in $x \in \mathbb{R}^d$, with periodic interval $P = \Pi_{i=1}^d [0, p_i] \subseteq \mathbb{R}^d$ and periodic initial data:

$$u_0(x + p_i \mathbf{e}_i) = u_0(x), \qquad i = 1,\dots,d. \tag{4.1}$$

The following result of [9] may be obtained as a consequence of Theorem 2.1, observing that (3.7) implies

$$\partial_t \alpha(u,\bar{u}) + \sum \partial_{x_k} \beta^k(u,\bar{u}) \leq 0, \tag{4.2}$$

in the sense of distributions, from which we easily verify (2.5) with $\psi(x,t)$ replaced by $u(x,t)$, where $\alpha(u,\bar{u})$ is the Dafermos' quadratic entropy associated with a strictly convex entropy $\eta(u)$ by

$$\alpha(u,\bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u-\bar{u}),$$

with associated entropy-fluxes $\beta^k(u,\bar{u})$ given by

$$\beta^k(u,\bar{u}) = q^k(u) - q^k(\bar{u}) - \nabla\eta(\bar{u})(f^k(u) - f^k(\bar{u})).$$

Theorem 4.1. *Assume that* $u(x,t) \in L^\infty(\mathbb{R}_+^{d+1})$ *is a periodic solution of* (3.3)-(3.2) *and that* $u^T(x,t)$ *is compact in* $L^1_{loc}(\mathbb{R}_+^{d+1})$. *Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T M_x(|u(x,t)-\bar{u}|)\,dt, \tag{4.3}$$

where

$$\bar{u} \equiv \frac{1}{|P|}\int_P u_0(x)\,dx. \tag{4.4}$$

Moreover, if system (3.3) *is endowed with a strictly convex entropy* η, *then*

$$\lim_{t\to\infty} \frac{1}{|P|}\int_P |u(x,t)-\bar{u}|dx = 0, \tag{4.5}$$

We now describe some of the most important applications of Theorem 4.1, given in [9]. We emphasize that many other examples are given in [9].

4.1 Multidimensional Scalar Conservation Laws

An important application of Theorem 4.1 given in [9] is concerned with multidimensional scalar conservation laws with periodic initial data. The existence of global entropy solutions of (3.3)-(3.2), in this case, when $u_0 \in L^\infty(\mathbb{R}^d)$, was proved by Kruzkov [42] by improving an earlier result of Volpert [61] for $u_0 \in BV(\mathbb{R}^d)$. On the other hand, compactness of uniformly bounded sequences of entropy solutions of such equations was proved by Lions, Perthame and Tadmor [45], for flux functions satisfying

$$\text{meas}\left\{ v \in \mathbb{R} \mid \tau + f'(v)\cdot k = 0 \right\} = 0, \tag{4.6}$$

$$\text{for any } (\tau,k) \in \mathbb{R}\times\mathbb{R}^d, \text{ with } \tau^2 + |k|^2 = 1. \tag{4.7}$$

A direct proof can be found in [12]. We then get as a consequence of Theorem 4.1 the following result of [9].

Theorem 4.2. *Let $u(x,t)$ be an entropy solution of* (3.3)-(3.2) *in $\mathbb{R}_+^{d+1}$ with periodic data $u_0(x)$ and periodic interval P. Assume that the condition* (4.6) *holds. Then*

$$\lim_{t\to\infty}\frac{1}{|P|}\int_P |u(x,t)-\bar{u}|\,dx = 0.$$

4.2 Equations of nonlinear elasticity

Let us consider the system of nonlinear elasticity equations

$$\begin{cases} \partial_t\tau - \partial_x v = 0, \\ \partial_t v - \partial_x\sigma(\tau) = 0, \end{cases} \tag{4.8}$$

where $\sigma(\tau) \in C^2$ satisfies $\sigma'(\tau) > 0$, $\tau \in \mathbb{R}$, and $\tau\sigma''(\tau) > 0$, for $\tau \neq 0$. Existence of L^∞ entropy solutions of (4.8)-(3.2), with initial data belonging to $L^\infty(\mathbb{R})$, was proved by DiPerna in [24]. From the latter also follows the compactness of uniformly bounded entropy solutions of (4.8)-(3.2). When the initial data are periodic the solutions obtained are periodic in x, with the same period for each $t > 0$. Hence, application of Theorem 4.1 gives the following result of [9].

Theorem 4.3. *Let $(\tau(x,t), v(x,t)) \in L^\infty(\mathbb{R}_+^2)$ be a periodic entropy solution of the equations of elasticity* (4.8) *with periodic interval P. Then $(\tau(x,t), v(x,t))$ asymptotically decays to*

$$(\bar\tau,\bar v) = \left(\frac{1}{|P|}\int_P \tau_0(x)dx,\ \frac{1}{|P|}\int_P v_0(x)dx\right),$$

in the sense of (4.5), *with $u(x,t) = (\tau(x,t), v(x,t))$.*

4.3 Isentropic Euler Equations

Consider the isentropic Euler equations for compressible fluids:

$$\begin{cases} \partial_t\rho + \partial_x m = 0, \\ \partial_t m + \partial_x(\frac{m^2}{\rho} + p(\rho)) = 0, \end{cases} \tag{4.9}$$

where ρ, m, and p are the density, the momentum, and the pressure, respectively. In a non-vacuum state ($\rho \neq 0$), $v = m/\rho$ is the velocity. The pressure $p(\rho)$

is a given function of the density ρ depending on compressible fluids under consideration. For the polytropic case, $p(\rho) = k^2\rho^\gamma, \gamma > 1$.

Consider the Cauchy problem for (4.9) with initial data

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)), \quad 0 \le \rho_0(x) \le C_0, \ |\frac{m_0(x)}{\rho_0(x)}| \le C_0 < \infty. \tag{4.10}$$

Existence of entropy solutions in L^∞, of (4.9), in the polytropic case ($p(\rho) = k^2\rho^\gamma$, $\gamma > 1$), with $(\rho_0, m_0) \in L^\infty(\mathbb{R})$, satisfying (4.10), was first proved by DiPerna [25], in the case $\gamma = 1 + 1/(2m+1), m \ge 2$ integer. His analysis was extended by Ding-Chen-Luo and Chen [23, 7], for $1 < \gamma \le 5/3$. Using a kinetic formulation for entropy inequalities associated with (4.9), Lions-Perthame-Tadmor [46] and Lions-Perthame-Souganidis [47] succeeded to obtain the same result for $\gamma \ge 3$ and $5/3 < \gamma < 3$, respectively. More recently, Chen-LeFloch [15] extended the analysis in [47] for more general pressure laws. The just referred results also give the compactness of uniformly bounded sequences of entropy solutions of (4.9), in the corresponding cases. When the initial data are periodic the solutions obtained are periodic in x, with the same period for each $t > 0$. Application of Theorem 4.1 then gives the following result of [9].

Theorem 4.4. *Let* $(\rho(x,t), m(x,t)), 0 \le \rho(x,t) \le C, |m(x,t)/\rho(x,t)| \le C$, *be a periodic entropy solution of* (4.9)-(4.10) *with periodic interval* P. *Then* $(\rho(x,t), m(x,t))$ *asymptotically decays to* $\frac{1}{|P|}\int_P(\rho_0(x), m_0(x))dx$ *in the sense of* (4.5).

5 Periodic solutions constructed through Glimm scheme

In [32] the construction of globally defined periodic entropy solutions of systems of conservation laws is achieved using a periodic version of the Glimm scheme. Such a periodic version for the Glimm scheme is not possible in general, if one has to follow the same procedures originally introduced by Glimm [37] in order to prove the uniform boundedness of the total variation of the approximate solutions. The main obstruction is that the functional used by Glimm contains a quadratic part, consisting of products of strengths of approaching waves, which has no periodic equivalent: waves that are approaching in a certain interval of one period may not be approaching in another interval of one period. In other words, in the cylinder $S^1 \times [0, \infty)$, any wave is approaching to any other wave of a different family. On the other hand, the notion of approaching waves in the whole line seems to be essential in the process of bounding the effects of wave interactions, in general. Nevertheless, as we describe below, starting with Nishida [49], there have been found many 2×2 systems, and even some very special $n \times n$ systems, with $n > 2$, for which it is possible to prove the uniform boundedness of the total variation of the approximate solutions using functionals which are linear in the wave strengths, that is, which contain no products of wave strengths. These classes include the systems of isentropic gas dynamics. For them, a periodic version of the Glimm scheme is then possivel as we will show. This relatively simple observation allows one to construct globally defined uniformly bounded periodic entropy solutions, which in many cases was not known. Then, using Theorem 4.1 combined with a compactness theorem in [24], one can prove the decay of these solutions, thus solving some important open problems. We next pass to a more precise description of our results.

In [49], Nishida proved the global existence of weak solutions of the Cauchy problem for the system

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + (1/v)_x &= 0, \end{aligned}$$

constructed by an adaptation of Glimm's method, for any bounded initial data belonging to $BV_{loc}(\mathbb{R})$, assuming values in the region $v > \delta > 0$. According to Nishida's theorem, the L^∞ norm of these weak solutions may increase unboundedly with time. His result was later extended by Bakhvalov [1], who identified a class of 2×2 systems

$$U_t + F(U)_x = 0, \tag{5.1}$$

with $U = (u_1, u_2)$, $F(U) = (f_1(U), f_2(U))$, to which Nishida's reasonings could be applied. The systems in the class determined by Bakhvalov are characterized by some conditions satisfied by the image of the corresponding shock curves in a plane of Riemann invariants. Here we will not give a description of Bakhvalov's conditions but instead we will consider the main representative of this class:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \end{cases} \tag{5.2}$$

where p is a smooth function defined in $v > 0$ satisfying

$$p'(v) < 0, \quad p''(v) > 0, \quad 3(p''(v))^2 \leq 2p'(v)p'''(v), \quad \text{for } v > 0. \tag{5.3}$$

For instance, $p(v) = \gamma v^{-\gamma}$, with $-1 < \gamma \leq 1$, $\gamma \neq 0$. It is not difficult to show that if p_1 and p_2 satisfy (5.3) then $C_1 p_1 + C_2 p_2$ satisfies (5.3), for any pair of nonnegative constants $(C_1, C_2) \neq (0, 0)$.

Set $U = (v, u)$ and

$$z(U) \equiv u + \int^v \sqrt{-p'(v)}\, dv, \qquad w(U) \equiv u - \int^v \sqrt{-p'(v)}\, dv. \tag{5.4}$$

Consider the initial condition

$$U(x, 0) = U_0(x), \qquad x \in \mathbb{R}, \tag{5.5}$$

with

$$0 < \delta \leq v_0(x) \leq M, \qquad |u_0(x)| \leq M, \tag{5.6}$$

for some positive constants δ, M. Suppose that

$$\mathrm{TV}\,(U_0|[0, \ell)) < \infty, \tag{5.7}$$

and

$$U_0(x + \ell) = U_0(x). \tag{5.8}$$

If

$$\Phi(v) = -\int^v \sqrt{-p'(v)}\, dv \nrightarrow +\infty, \qquad \text{as } v \to 0, \tag{5.9}$$

which is the case, for instance, when $p(v) = \gamma v^{-\gamma}$, with $-1 < \gamma < 1$, $\gamma \neq 0$, we further assume that

$$w_0 = \sup_{x \in \mathbb{R}} w(U_0(x)) < z_0 = \inf_{x \in \mathbb{R}} z(U_0(x)). \tag{5.10}$$

Concerning the problem (5.2)-(5.8), the following result is proved in [32]. Let

$$\bar{U} = \frac{1}{\ell} \int_0^\ell U_0(x)\, dx. \tag{5.11}$$

Theorem 5.1. *There exists a global periodic entropy solution of* (5.2)-(5.10), *which belongs to* $L^\infty \cap BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$. *This solution* $U(x,t)$ *also satisfies* $w(U(x,t)) \le w_0$, $z(U(x,t)) \ge z_0$. *Furthermore, it has the following decay property*

$$\operatorname{ess} \lim_{t\to\infty} \int_0^\ell |U(x,t) - \bar{U}|\, dx = 0. \tag{5.12}$$

We recall that the problem of the decay of the large data solutions obtained by Nishida was proposed in [38] and remained open since then. On passing from Lagrangian to Eulerian coordinates, Theorem 5.1 gives a global periodic entropy solution in $L^\infty \cap BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$ of the corresponding problem for the system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \big(\rho u^2 + P(\rho)\big)_x = 0, \end{cases} \tag{5.13}$$

where $\rho = v^{-1}$ and $P(\rho) = p(1/\rho)$.

An important fact concerning Bakhvalov's conditions was noticed by DiPerna [27]. Namely, he proved that if in (5.2) one has $p'(v) < 0$, $p''(v) > 0$, for $v > 0$, and

$$\lim_{v\to 0} v^{\alpha+j} \frac{d^j}{dv^j} p(v) \neq 0, \ \lim_{v\to\infty} v^{\beta+j} \frac{d^j}{dv^j} p(v) \neq 0, \ \alpha, \beta > 1, \ j = 0, \dots, 5, \tag{5.14}$$

then, given any $\bar{U} \in \{(u,v) \ : \ v > 0\}$, there exist Riemann invariants, $z' = \phi(z)$, $w' = \psi(w)$, and a neighborhood $\mathcal{V} = \mathcal{V}(\bar{U})$ such that the image in the (z', w')-plane of the segments of shock curves contained in $\mathcal{V}$ satisfy Bakhvalov's conditions. As a consequence, DiPerna proves the global existence of an entropy solution of the Cauchy problem, provided that the total variation of the initial data is less than $C(p)\tilde{\zeta}$, where $C(p)$ is a constant depending only on the nonlinear function $p(v)$, and $\tilde{\zeta} = \lim_{x\to\infty}\{w(U_0(x)) - z(U_0(x))\}$. In particular, for $p(v) = v^{-\gamma}$, $1 \le \gamma < +\infty$, which clearly satisfies (5.14), one has $C(p) \ge C_0$, for some fixed constant $C_0 > 0$, and $\tilde{\zeta} = \dfrac{1}{\gamma - 1} \lim_{x\to+\infty} v_0^{(1-\gamma)}(x)$. In other words, the restriction on the magnitude of the total variation of the initial data is $O((\gamma-1)^{-1})$. This shows that the closer γ is to 1 the larger the total variation of the initial data can be. For these systems of isentropic gas dynamics, a similar result had also been obtained by Nishida and Smoller [50] through different means. Concerning the systems (5.2) satisfying the conditions (5.14), using our periodic formulation based on DiPerna's analysis, we have the following result of [32]. Set $\zeta(U) = w(U) - z(U)$, and $\bar{\zeta} = \zeta(\bar{U})$.

Theorem 5.2. *Consider the problem* (5.2)-(5.10) *with* $p(v)$ *satisfying* (5.14). *Let* $\bar{U}$ *be as above. There exists and a constant* $C(p)$, *depending only on* p, *such that if*

$$TV(U_0|[0,\ell)) \leq C(p)\bar{\zeta}, \tag{5.15}$$

then there exists a global periodic entropy solution of (5.2)-(5.10), *which belongs to* $L^\infty \cap BV_{loc}(\mathbb{R}\times\mathbb{R}_+)$. *This solution* $U(x,t)$ *also satisfies* $w(U(x,t)) \leq w_0$, $z(U(x,t)) \geq z_0$. *Furthermore, it has the decay property* (5.12).

A class of systems identified by DiPerna [26] is also considered in [32], with similar properties to that studied in [1]. We will not give a description of DiPerna's class but instead we will consider the main representative of this class:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + \left(\frac{1}{2}u^2 + \frac{\kappa^2}{\gamma-1}\rho^{\gamma-1}\right)_x = 0, \quad \kappa > 0,\ 1 < \gamma < 3, \end{cases} \tag{5.16}$$

which is motivated by the isentropic gas dynamics equations for a polytropic gas. Let $U = (\rho, u)$, $F(U) = \frac{1}{2}u^2 + \frac{\kappa^2}{\gamma-1}\rho^{\gamma-1}$, $\rho > 0$, and

$$z(U) = u - \frac{\kappa}{\beta}\rho^\beta, \qquad w(U) = u + \frac{\kappa}{\beta}\rho^\beta, \quad \beta = \frac{1}{2}(\gamma - 1). \tag{5.17}$$

Denote $v = \rho^{-1}$ and consider the initial conditions (5.5)-(5.8) and

$$z_0 = \sup_{x\in\mathbb{R}} z(U_0(x)) < w_0 = \inf_{x\in\mathbb{R}} w(U_0(x)). \tag{5.18}$$

Concerning the system (5.16) we also have the following result proved in [32].

Theorem 5.3. *There exists a global entropy solution of* (5.16), (5.5)-(5.8), (5.18), *which belongs to* $L^\infty \cap BV_{loc}(\mathbb{R}\times\mathbb{R}_+)$ *and is periodic with period* ℓ. *This solution* $U(x,t)$ *also satisfies* $z(U(x,t)) \leq z_0$, $w(U(x,t)) \geq w_0$. *Moreover, it accomplishes the decay property* (5.12).

In Theorems 5.1–5.3, the global boundedness of the periodic solutions is obtained by means of a periodic version of Nishida's modification of the Glimm scheme. The decay property (5.12) is then a consequence of a combination of Theorem 4.1 and a compactness theorem in [24].

In connection with the above theorems, we recall that a theorem in [38] establishes the existence and decay of periodic solutions for a class of strictly hyperbolic genuinely nonlinear 2×2 systems, including (5.2), with $p'(v) < 0$, $p''(v) > 0$, for $v > 0$. The solutions in [38] are obtained by the Glimm method for initial data in L^∞ but with very small oscillation. A careful analysis of the asymptotics and the mechanism generating the asymptotic patterns for solutions having the decay property proved in [38] is provided in [22]. Although the classes of systems we consider are slightly less general and the initial data are in BV_{loc}, in many cases our results allow initial data with large oscillation. Finally, we would like to remark that a result analogous to Theorems 5.1 and 5.3 is obtained similarly for the relativistic Euler equations studied by Smoller and Temple in [57], based on a periodic version of the analysis made therein.

The above results prompt us with a possible strategy for solving the longstanding problem of the global existence of periodic solutions of the 3×3 compressible Euler equations:

$$u_t + p_x = 0, \tag{5.19}$$

$$\tau_t - u_x = 0, \tag{5.20}$$

$$(e + \frac{u^2}{2})_t + (pu)_x = 0, \tag{5.21}$$

$$S_t \geq 0, . \tag{5.22}$$

Here, u, τ, p, e, S represent the velocity, specific volume, pressure, internal energy and entropy, respectively. Another important variable is the temperature θ. As usual, we may assume that p, e, θ are given functions of (τ, S), so $p = p(\tau, S)$, $e = e(\tau, S)$, $\theta = \theta(\tau, S)$, where these functions should be compatible with the second law of thermodynamics,

$$de = \theta\, dS - p\, d\tau. \tag{5.23}$$

For concreteness, we restrict our discussion to ideal polytropic gases where $p = R\theta\tau^{-1}$, $\theta = c_v e$, for positive constants c_v, R, and so we have, for some $\kappa > 0$,

$$p(\tau, S) = \kappa \exp(S/c_v)\tau^{-\gamma},\ e(\tau, S) = \frac{c_v\kappa}{R}\exp(S/c_v)\tau^{-\gamma+1},\ \gamma = 1 + \frac{R}{c_v} > 1.$$

Equations (5.19)-(5.21) may then be rewritten in the form

$$u_t + (\kappa\Sigma\,\tau^{-\gamma})_x = 0, \tag{5.24}$$

$$\tau_t - u_x = 0, \tag{5.25}$$

$$\left(\frac{c_v\kappa}{R}\Sigma\,\tau^{-\gamma+1} + \frac{u^2}{2}\right)_t + (\kappa\Sigma\,\tau^{-\gamma}u)_x = 0, \tag{5.26}$$

if we define $\Sigma = \exp(S/c_v)$. System (5.19)-(5.21) has three characteristic families corresponding to the eigenvalues $\lambda_1 = -\sqrt{-p_\tau}$, $\lambda_2 \equiv 0$, $\lambda_3 = \sqrt{-p_\tau}$. The first and third family are genuinely nonlinear while the second is linearly degenerate according to [44]. This suggests a decoupling of the first and third families from the second one. Since S is a Riemann invariant for the first and third families, this means that we should consider u, τ separately from S. The system put in the form (5.24)-(5.26) is very suggestive concerning this decoupling. So let be given periodic initial data, with bounds in the total variation per period and L^∞ norm to be suitably chosen. The first step would be to construct approximations u^h, τ^h, for u, τ, using a periodic formulation of the Glimm scheme, as it is done for the proof of Theorem 5.2. Simultaneously, we construct an approximation Σ^h, for Σ, from the same scheme, but defining Σ^h at a sucessive time step as the average of Σ^h over the top side of the preceding mesh rectangle. The second step is to try to prove that the space total variation per period of (u^h, τ^h) is uniformly bounded in time. The third step is to observe that the increase in the L^∞ norm of Σ^h can only be due to the crossing of shock waves of the first and third families. Since, the jump of Σ^h across (small) shocks is of the order of the cube of the strength of the shock, we would obtain an uniform L^∞ bound for Σ^h from the uniform boundedness of the total variation of (u^h, τ^h). In this way we would be able to prove the strong convergence of (u^h, τ^h) and the weak star convergence of Σ^h to a solution (u, τ, Σ) of (5.24)-(5.26), from which (5.19)-(5.21) can be recovered. This solution would further satisfy

$$\Sigma_t \geq 0, \tag{5.27}$$

which is not equivalent to (5.22) but still serves as an entropy inequality. This strategy is currently under investigation.

6 Decay of almost periodic solutions

We are interested in the asymptotic behavior of solutions $u(x,t)$ of (3.1)-(3.2) which are generalized almost periodic functions, in the sense of Stepanoff-Wiener, which we abridge by saying that $u(x,t)$ is S-$a.p.$, in the x variable, locally uniformly in $t \geq 0$. For definitions and basic properties about generalized almost periodic functions see section 2.

As in [9], we denote by $u^T(x,t)$, $T > 0$, the scaling sequence associated with $u(x,t)$ defined by

$$u^T(x,t) = u(Tx, Tt). \tag{6.1}$$

Set

$$\bar{u} = \lim_{L\to\infty} \frac{1}{(2L)^d} \int_{|x|_\infty \le L} u_0(x)\, dx. \tag{6.2}$$

The following result of [31] maybe obtained as a corollary of Theorem 2.1, observing that (3.6) implies

$$\partial_t \alpha(u, \bar{u}) + \sum \partial_{x_k} \beta^k(u, \bar{u}) \le \sum \partial^2_{x_k x_l} \gamma_{kl}(u, \bar{u}), \tag{6.3}$$

in the sense of distributions, from which we easily verify (2.5) with $\psi(x,t)$ replaced by $u(x,t)$, where $\alpha(u, \bar{u})$ is the Dafermos' quadratic entropy associated with a strictly convex entropy $\eta(u)$ by

$$\alpha(u, \bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u - \bar{u}),$$

with associated entropy-fluxes $\beta^k(u, \bar{u})$ and entropy-viscosities $\gamma_{kl}(u, \bar{u})$ given by

$$\begin{aligned} \beta^k(u, \bar{u}) &= q^k(u) - q^k(\bar{u}) - \nabla\eta(\bar{u})(f^k(u) - f^k(\bar{u})), \\ \gamma_{kl}(u, \bar{u}) &= b_{kl}(u) - b_{kl}(\bar{u}) - \nabla\eta(\bar{u})(a_{kl}(u) - a_{kl}(\bar{u})). \end{aligned}$$

Theorem 6.1. *Let $u(x,t)$ be a solution of* (3.1)-(3.2) *which is S-a.p. in x, locally uniformly in $t \ge 0$. Let $l_\varepsilon(t)$ denote an inclusion interval of $u(x,t)$ with respect to $\varepsilon > 0$. Assume the following:*

(i) $l_\varepsilon(t)/t \to 0$ as $t \to \infty$;

(ii) $u^T(x,t)$ is pre-compact in $L^1_{\text{loc}}(\mathbb{R}^{d+1}_+)$ as $T \to \infty$.

Then $u^T \to \bar{u}$ as $T \to \infty$ in $L^1_{\text{loc}}(\mathbb{R}^{d+1}_+)$ and one has

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T M_x(|u(x,t) - \bar{u}|)\, dt = 0. \tag{6.4}$$

Moreover, in the inviscid case where $a_{kl}(u) \equiv 0$, for all k, l, if (3.1) *is endowed with a strictly convex entropy then one has $u(\xi t, t) \to \bar{u}$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, as $t \to \infty$. In particular, one has*

$$M_x(|u(x,t) - \bar{u}|) \to 0 \qquad \text{as } t \to \infty. \tag{6.5}$$

The latter holds also in the viscous case, $a_{kl}(u) \not\equiv 0$, for some k, l, provided that (3.1) *is endowed with a strictly convex entropy satisfying* (3.5) *and $\nabla_x u(x,t)$ is uniformly bounded in $\mathbb{R}^d \times [t_0, \infty)$ for some $t_0 > 0$.*

Remark 6.1. The first part of the statement of Theorem 6.1 holds also for almost periodic solutions (in a suitable sense) of the more general class of viscous systems of the form

$$\partial_t u + \sum_k \partial_{x_k} f^k(u) = \sum_{k,l} \partial_{x_k} \left(B_{kl}(u) \partial_{x_l} u \right), \tag{6.6}$$

as it is clear from the proof. The second part also holds in this cotext, provided that we define the notion of a strictly convex entropy η for (6.6) to mean now that η is strictly convex, there exist functions q^k such that

$$\nabla q^k(u) = \nabla \eta(u) \nabla f^k(u),$$

and

$$\sum_{k,l} \nabla^2 \eta(u)(B_{kl}(u) v_k, v_l) \geq 0, \quad \text{for all } (v_1, \dots, v_d) \in (\mathbb{R}^n)^d.$$

7 Almost periodic solutions of scalar conservation laws in several space variables

We consider the initial value problem for a scalar conservation law in several space variables

$$\partial_t u + \sum_{k=1}^{d} \partial_{x_k} f^k(u) = 0, \tag{7.1}$$

$$u(x,0) = u_0(x), \tag{7.2}$$

where the $f^k(u)$ are smooth functions and u_0 is a bounded S-$a.p.$ function defined in $\mathbb{R}^d$. We apply Theorem 6.1 to obtain the decay of the entropy solution of (7.1)-(7.2) provided that u_0 satisfies a suitable condition on the growth of its inclusion intervals $l_\varepsilon(0)$ as $\varepsilon \to 0$. Existence and L^1_{loc} stability of entropy solutions of (7.1)-(7.2), with $u_0 \in L^\infty(\mathbb{R}^d)$, was proved by Kruzkov [42].

The following theorem of [31] is obtained as a consequence of Theorem 6.1, the stability result in [42], and the compactness result in [45].

Theorem 7.1. *Let $f(u) = (f^1(u), \dots, f^d(u))$ satisfy the nonlinearity condition*

$$meas\{u \in \mathbb{R} : \tau + \kappa \cdot f'(u) = 0\} = 0, \quad \forall (\tau, \kappa) \in \mathbb{R}^{d+1}, s.t.\ \tau^2 + |\kappa|^2 = 1. \tag{7.3}$$

Assume $u_0 \in L^\infty(\mathbb{R}^d)$ is S-a.p. and there exists a sequence of S-a.p. functions $u_{0,\nu}$, $\nu \in \mathbb{N}$, belonging to $L^\infty(\mathbb{R}^d)$ such that $M_x(|u_0(x) - u_{0,\nu}(x)|) \to 0$, as $\nu \to \infty$, and so that, for each $\nu \in \mathbb{N}$, the inclusion intervals of $u_{0,\nu}$ with respect to ε, $l_\varepsilon^\nu(0)$, satisfy $\varepsilon^{1/d} l_\varepsilon^\nu(0) \to 0$, as $\varepsilon \to 0$. Let $u(x,t)$ be the unique entropy solution of (7.1)-(7.2). Then $u(\xi t, t) \to \bar{u}$ as $t \to \infty$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, and, in particular, $M_x(|u(x,t) - \bar{u}|) \to 0$ as $t \to \infty$.

Remark 7.1. Clearly, the hypothesis of Theorem 7.1 concerning the initial data $u_0(x)$ is satisfied by any generalized limit periodic function belonging to $L^\infty(\mathbb{R}^d)$, that is, any S-$a.p.$ function in $L^\infty(\mathbb{R}^d)$ which is limit of L^∞ purely periodic functions in the sense of the norm $\|\psi\|_W = M_x(|\psi|)$, for $\psi \in S$-$a.p.$.

8 Almost periodic solutions of inviscid systems in chromatography

In this section we analyze the application of Theorem 6.1 to some special inviscid systems of conservation laws for which the compactness of the solution operator and the L^1 stability with respect to initial data have been proved in recent works. Namely, we are going to consider the initial value problem

$$\partial_t u + \partial_x f(u) = 0, \tag{8.1}$$

$$u(x,0) = u_0(x), \tag{8.2}$$

where (8.1) is the $n \times n$ chromatography system. The analysis here is very similar to the case of scalar conservation laws analized in section 7. For this system one has

$$f_i(u) = \frac{k_i u_i}{1 + u_1 + \cdots + u_n}, \qquad i = 1, \ldots, n, \tag{8.3}$$

where k_i are given numbers with

$$0 < k_1 < k_2 < \cdots < k_n.$$

These systems belong to the so called Temple class which is characterized by the following two properties: (1) There exists a complete set of Riemann invariants defined everywhere in the domain of f, $\mathcal{U} \subseteq \mathbb{R}^n$, that is, a set of functions $\{\omega_1(u), \ldots, \omega_n(u)\}$ satisfying $\nabla\omega_i(u) = l_i(u)$, where the $l_i(u)$ are n linearly independent left eigenvectors of $\nabla f(u)$, $i = 1, \ldots, n$; (2) the level sets $\{u \in \mathcal{U} : \omega_i(u) = \text{constant}\}$ are hyperplanes (*cf.* [53]). Recently, Bressan and Goatin [6] constructed a continuous semigroup of solutions on a domain of L^∞ functions, for systems (8.1) in the Temple class, which are strictly hyperbolic and *genuinely nonlinear*, where the trajectories depend Lipschitz continuously on the initial data in the L^1 metric. In [6], the initial data are supposed to take values in a domain $E \subseteq \mathcal{U}$ of the form

$$E = \{u \in \mathcal{U} \,:\, \omega_i(u) \in [a_i, b_i],\ i = 1, \dots, n\},$$

in which the following strong hyperbolicity condition holds:

(SH) Given any n vectors $u^1, \dots, u^n \in E$, the eigenvalues $\lambda_1(u), \dots, \lambda_n(u)$ of $\nabla f(u)$ at these points are such that $\lambda_1(u^1) < \lambda_2(u^2) < \cdots < \lambda_n(u^n)$. Moreover, the right eigenvectors $r_1(u^1), r_2(u^2), \dots, r_n(u^n)$ are linearly independent.

As remarked in [6] the above assumption is automatically satisfied if the system is strictly hyperbolic and E is contained in a small neighborhood of a given point. Concerning compactness of the solution operator of (8.1),(8.2), (8.3), we recall that this has been proved by James, Peng and Perthame [41], where the compactness is achieved through compensated compactness [59, 48, 24] and a kinetic formulation for the chromatography system. So, combining the L^1 stability theorem in [6], the compactness theorem in [41], and Theorem 6.1 we get the following result of [31].

Theorem 8.1. *Consider the problem* (8.1), (8.2), (8.3). *Assume* (8.1), (8.3) *is strictly hyperbolic and genuinely nonlinear and that* $u_0 \in L^\infty(\mathbb{R}^d)$ *is S-a.p.. Suppose that there exists a sequence of S-a.p. functions* $u_{0,\nu}$, $\nu \in \mathbb{N}$, *belonging to* $L^\infty(\mathbb{R}^d)$ *such that* $M_x(|u_0(x) - u_{0,\nu}(x)|) \to 0$, *as* $\nu \to \infty$, *and so that, for each* $\nu \in \mathbb{N}$, *the inclusion intervals of* $u_{0,\nu}$ *with respect to* ε, $l_\varepsilon^\nu(0)$, *satisfy* $\varepsilon l_\varepsilon^\nu(0) \to 0$, *as* $\varepsilon \to 0$. *Suppose also that* u_0 *and all* $u_{0,\nu}$ *take their values in a region E where (SH) is satisfied. Let* $u(x,t)$ *be the unique entropy solution of* (7.1)-(7.2). *Then* $u(\xi t, t) \to \bar{u}$ *as* $t \to \infty$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, *and, in particular,* $M_x(|u(x,t) - \bar{u}|) \to 0$ *as* $t \to \infty$.

9 Almost periodic solutions of inviscid systems in isentropic gas dynamics

Here we consider the application of Theorem 6.1 to the relativistic isentropic Euler equation, which is a 2×2 system of the form (8.1) with

$$(u_1, u_2) = \Big(\rho \frac{1 + (\zeta^2 v^2)/c^4}{1 - v^2/c^2}, \rho v \frac{1 + (\zeta^2 v^2)/c^4}{1 - v^2/c^2}\Big), \tag{9.1}$$

and

$$f(u_1, u_2) = \big(\rho v \frac{1 + (\zeta^2 v^2)/c^4}{1 - v^2/c^2}, \rho \frac{v^2 + \zeta^2}{1 - v^2/c^2}\big), \tag{9.2}$$

where ζ, c are positive constants representing the sound and light speed, respectively, ρ is the density and v is the velocity of the gas. We observe that in the limit $c \to \infty$ (9.1)-(9.2) reduce to the classical Euler isentropic gas dynamics model for a polytropic gas with $\gamma = 1$, that is,

$$(u_1, u_2) = (\rho, \rho v), \qquad f(u_1, u_2) = \big(\rho v, \rho(v^2 + \zeta^2)\big). \tag{9.3}$$

In [19], Colombo and Risebro prove the existence of an L^1-Lipschitz continuous semigroup S, defined on functions of bounded variation, with total variation not necessarily small, whose trajectories are weak entropy solutions of (8.1),(9.1),(9.2). Given S-$a.p.$ initial data in $BV_{loc}(\mathbb{R})$, we may apply the existence and stability result in [19] to obtain the global existence of an entropy weak solution, which is S-$a.p.$ in x for each fixed t. The growth of the inclusion intervals $l_\varepsilon(t)$, as $t \to \infty$, is again determined by the growth of the initial inclusion intervals $l_\varepsilon(0)$ as $\varepsilon \to 0$, but now not in an explicit way. Nevertheless, we may deduce the existence of a family of functions $H_\lambda : (0, \infty) \to (0, \infty)$, $\lambda > 0$, satisfying $H_\lambda(s) \to \infty$ as $s \to 0+$, such that, if $l_\varepsilon(0)/H_\lambda(\varepsilon) \to 0$ as $\varepsilon \to 0$, for any fixed $\lambda > 0$, then $l_\varepsilon(t)/t \to 0$ as $t \to \infty$, for each fixed $\varepsilon > 0$. The functions H_λ are related with the growth of $TV(u_0|(-s, s))$ as $s \to \infty$. So, now the restriction on the initial data appears as a correlation between the growth rate of the inclusion intervals as $\varepsilon \to 0$ and the growth rate of the total variation over the intervals $(-s, s)$ as $s \to \infty$.

So, combining the L^1 stability theorem in [19], the compactness theorem in [24], and Theorem 6.1 we arrive at the following result.

Theorem 9.1. *Consider the problem* (8.1), (8.2),(9.1), (9.2). *Assume* $u_0 \in BV_{loc}(\mathbb{R})$ *is* S-$a.p.$. *Then there exists a global weak entropy solution of this problem, which is* S-$a.p.$ *in* x *for each* $t > 0$. *Also, there is a family of functions* $H_\lambda : (0, \infty) \to (0, \infty)$, $\lambda > 0$, *satisfying* $H_\lambda(s) \to \infty$ *as* $s \to 0+$, *such that, if* $l_\varepsilon(0)/H_\lambda(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *for any fixed* $\lambda > 0$, *then* $l_\varepsilon(t)/t \to 0$ *as* $t \to \infty$, *for each fixed* $\varepsilon > 0$. *In particular, if* $l_\varepsilon(0)/H_\lambda(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *for any fixed* $\lambda > 0$, *then* $M_x(|u(x,t) - \bar{u}|) \to 0$ *as* $t \to \infty$.

10 Almost periodic solutions of viscous systems of conservation laws in several space variables

Here we consider viscous systems of conservation laws of the form

$$\partial_t u + \sum_{k=1}^{d} \partial_{x_k} f^k(u) = \Delta u, \qquad t > 0,\ x = (x_1, \dots, x_d) \in \mathbb{R}^d, \tag{10.1}$$

where $u(x,t)$ assumes values in a domain $\mathcal{U} \subseteq \mathbb{R}^n$, $f^k : \mathcal{U} \to \mathbb{R}^n$ are smooth functions, $k = 1, \dots, d$ and Δ denotes the Laplacian operator in $\mathbb{R}^d$. Let be given initial data

$$u(x,0) = u_0(x), \tag{10.2}$$

where $u_0 \in L^\infty(\mathbb{R}^d)$ is S-$a.p.$, and takes its values in a closed region $\overline{\Omega} \subseteq \mathcal{U}$ which is positively invariant under the flow generated by (10.1). Such regions were characterized in [18] and their existence is known for many particular systems (see examples in the next section). In the simplest case of scalar equations ($d = 1$) invariant compact intervals are obtained from the well known maximum principle.

For flux functions f^k, $k = 1, \dots, d$, which are smooth and Lipschitz continuous over the positively invariant closed region $\overline{\Omega}$, the existence and uniqueness of global smooth solutions of (10.1)-(10.2) is well known and can be constructed through the procedures in [39]. Concerning these solutions we have the following result of [31].

Theorem 10.1. *Let $\overline{\Omega}$ be a positively invariant closed region for* (10.1)*, f^k, $k = 1, \dots, d$, be smooth over $\mathcal{U} \supset \overline{\Omega}$ and Lipschitz continuous over $\overline{\Omega}$, and $u_0 \in L^\infty(\mathbb{R}^d)$ be S-a.p. assuming its values in $\overline{\Omega}$. Let $u(x,t)$ be the classical solution of* (10.1)-(10.2) *which is defined and smooth in $\mathbb{R}^d \times (0,\infty)$. Then $u(x,t)$ is S-a.p. in x, locally uniformly in $t \in [0,\infty)$, and its inclusion intervals with respect to $\varepsilon > 0$, $l_\varepsilon(t)$, satisfy $l_\varepsilon(t)/t \to 0$, as $t \to \infty$, provided the inclusion intervals of u_0, $l_\varepsilon(0)$, satisfy $(\log \varepsilon)^{-1} l_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. Moreover, if $\overline{\Omega}$ is bounded then, for any $t_0 > 0$, $\nabla_x u$ is uniformly bounded for $t \geq t_0$.*

Theorem 10.1 is applicable to the viscous pertubation of all hyperbolic systems of conservation laws for which the existence of compact positively invariant regions is known. If in addition the compactness of the scaling sequence u^T is known, an application of Theorem 6.1 immediately gives the decay of the solution of the perturbed viscous system. This includes, in particular, the viscous perturbations of all systems for which the decay of periodic solutions was obtained in [9]. We just mention a few examples below.

10.1 Viscous scalar conservation laws

If in (10.1) $n = 1$ then as is well known there exists a unique uniformly bounded solution of (10.1)-(10.2), $u(x, t)$, the uniform boundedness being consequence of the usual maximum principle. Hence, if the initial data $u_0(x)$ is S-$a.p.$ one obtains by Theorem 10.1 that $u(x, t)$ is S-$a.p.$ and $l_\varepsilon(t)/t \to 0$ as $t \to \infty$, provided $(\log \varepsilon)^{-1} l_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. Now, again using a compactness result in [45] we obtain the compactness of the scaling sequence $u^T(x, t)$ and so we can apply Theorem 6.1 to conclude the decay of $u(x, t)$ to $\bar{u}$, as $t \to \infty$, in particular that $M_x(|u(x, t) - \bar{u}|) \to 0$ as $t \to \infty$. We observe the curious fact that the restriction over the growth of the inclusion intervals of the initial data as $\varepsilon \to 0$ is stronger in this case than in the inviscid case.

10.2 Nonlinear elasticity with artificial viscosity

Consider the 2×2 one-dimensional system of nonlinear elasticity with artificial viscosity given by

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = \partial_x^2 u_1, \\ \partial_t u_2 - \partial_x \sigma(u_1) = \partial_x^2 u_2, \end{cases} \tag{10.3}$$

with $\sigma'(v) > 0$ and $v\sigma''(v) > 0$ if $v \neq 0$. As is well known (see, *e.g.*, [53]) this system admits a family of bounded positively invariant regions which may include any bounded set in $\mathbb{R}^2$. Using the principle of invariant regions in [18] one may, in a standard way, extend the unique local solution to a unique globally defined uniformly bounded solution of (10.3)-(10.2). Hence, Theorem 10.1 is applicable and one obtains that the solution is S-$a.p.$ and satisfies $l_\varepsilon(t)/t \to 0$ as $t \to \infty$ as long as $(\log \varepsilon)^{-1} l_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. Now, DiPerna's compactness theorem in [24] implies that the scaling sequence u^T is compact in $L^1_{\text{loc}}(\mathbb{R}^2_+)$. Again, we can apply Theorem 6.1 and obtain, in particular, $M_x(|u(x, t) - \bar{u}|) \to 0$ as $t \to \infty$. We obtain the same result for a number of other viscous systems which are also endowed with bounded positively invariant regions and for which compensated compactness has beeing successfully applied such as the $n \times n$ system of chromatography with Langmuir coordinates [41], the quadractic systems in [14], the conjugate type systems in [35], etc..

10.3 Isentropic gas dynamics with artificial viscosity

Let us consider the 2×2 one-dimensional system of isentropic gas dynamics, for ideal polytropic gases, with an artificial viscosity given by

$$\begin{cases} \partial_t \rho + \partial_x m = \partial_x^2 \rho, \\ \partial_t m + \partial_x(\frac{m^2}{\rho} + p(\rho)) = \partial_x^2 m, \quad p(\rho) = \kappa \rho^\gamma, \end{cases} \tag{10.4}$$

with $\gamma > 1$. This system is also endowed with a family of positively invariant regions given by $-C\rho + \rho \int^\rho (\sqrt{p'(\rho)}/\rho)\, d\rho \le m \le C\rho - \rho \int^\rho (\sqrt{p'(\rho)}/\rho)\, d\rho$, with $C > 0$ (*cf.* [25]). If the initial data satisfies $\rho_0(x) > \delta > 0$ and $m_0(x) \le C_0\rho_0(x)$, for some $C_0 > 0$, the existence of a unique local solution may be proved in a standard way; this solution then can be extended as long as $\rho(x,t) > 0$. As explained in [31], the proof that the vacuum ($\rho = 0$) is not assumed in finite time is then a decisive point for the global existence of a solution to (10.4)-(10.2). The proof of this property given in [25], which assumes square integrability of $(\rho_0 - \bar{\rho}, m)$, for a certain $\bar{\rho}$, is not adequate here since we want to consider almost periodic initial data. Nevertheless, the proof that ρ remains bounded away from vacuum given in [13] does not make use of square integrability of the initial data and can be easily adapted to the case of a Cauchy problem.

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